TRINITY COLLEGE

Semester One Examination, 2022

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 3

Section One: Calculator-free

SOLUTIONS

WA student number: In figures

In words

Your name

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	50	35
Section Two: Calculator-assumed	12	12	100	90	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (50 Marks)**

This section has **seven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(6 marks)**

The polynomial $f(z) = g(z) \times h(z)$, where $h(z) = z^2 - 4z + 5$.

(a) Show that $z - 2 - i$ is a factor of $h(z)$. (2 marks)

Solution
$h(2+i) = (2+i)^2 - 4(2+i) + 5$ $= 4 + 4i - 1 - 8 - 4i + 5$ $= 0$
Specific behaviours
✓ substitutes $h(2+i)$ ✓ fully expands all terms and then simplifies

(b) Given that $f(z) = z^4 - 6z^3 + 17z^2 - 26z + 20$, solve $f(z) = 0$, giving all solutions in Cartesian form. (4 marks)

Solution
Since $h(z)$ is a factor of $f(z)$ then $z = 2 + i$ and its conjugate $z = 2 - i$ will both be solutions. $f(z) = g(z)h(z)$ $z^4 - 6z^3 + 17z^2 - 26z + 20 = (z^2 + az + 4)(z^2 - 4z + 5)$ Comparing z coefficients, $-26 = 5a - 16 \Rightarrow a = -2 \Rightarrow g(z) = z^2 - 2z + 4$. $z^2 - 2z + 4 = 0$ $(z-1)^2 - 1 = -4$ $(z-1)^2 = 3i^2$ $z = 1 \pm \sqrt{3}i$ Hence $f(z) = 0$ when $z = 2 \pm i, z = 1 \pm \sqrt{3}i$.
Specific behaviours
✓ uses result from part (a) to state two solutions ✓ determines $g(z)$ ✓ one correct solution to $g(z) = 0$ ✓ states all correct solutions

Question 2 **(5 marks)**

(a) Express the complex number $\dfrac{6}{1+\sqrt{3}i}$ in the form $r\,\text{cis}\,\theta\,, -\pi < \theta \le \pi$. (3 marks)

Solution
$\dfrac{6}{1+\sqrt{3}i} \times \dfrac{1-\sqrt{3}i}{1-\sqrt{3}i} = \dfrac{6}{4}(1-\sqrt{3}i)$ $= 3\,\text{cis}\left(-\dfrac{\pi}{3}\right)$
Specific behaviours
✓ exposes real and imaginary parts   ✓ correct modulus   ✓ correct answer in polar form

(b) When $u = 2\,\text{cis}\left(\dfrac{3\pi}{10}\right)$ and $v = 3\,\text{cis}\left(-\dfrac{\pi}{8}\right)$ determine

(i) $|uv^2|$. (1 mark)

Solution
$2 \times 3^2 = 18$
Specific behaviours
✓ correct value

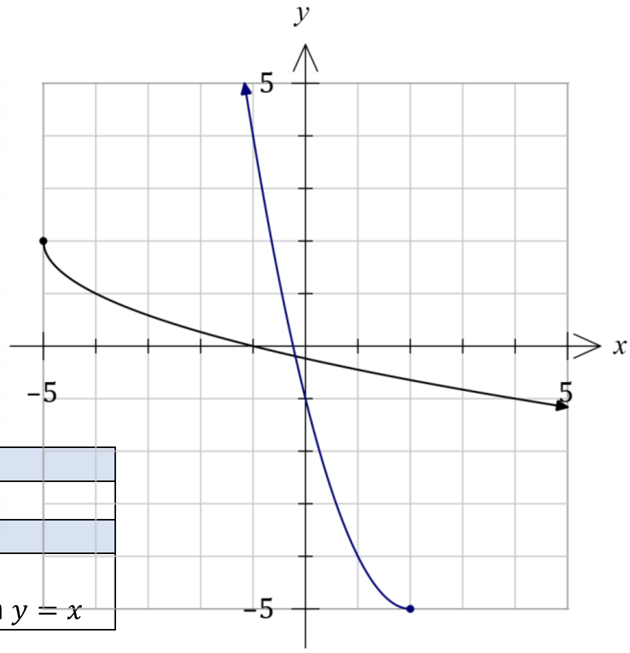
(ii) $\arg(u \div v)$. (1 mark)

Solution
$\dfrac{3\pi}{10} - \left(-\dfrac{\pi}{8}\right) = \dfrac{(12+5)\pi}{40} = \dfrac{17\pi}{40}$
Specific behaviours
✓ correct value

Question 3 **(8 marks)**

Function f is defined as $f(x) = 2 - \sqrt{x+5}$.

The graph of $y = f(x)$ is shown at right.



Solution (a)
See graph
Specific behaviours
✓ axis intercepts
✓ endpoint clearly reflection of $f(x)$ in $y = x$

(a) Sketch the graph of $y = f^{-1}(x)$ on the axes above. (2 marks)

(b) State the domain and range of $f^{-1}(x)$. (2 marks)

Solution
$D_{f^{-1}} = \{x: x \in \mathbb{R}, x \le 2\}$, $R_{f^{-1}} = \{y: y \in \mathbb{R}, y \ge -5\}$
Specific behaviours
✓ correct domain
✓ correct range

Function g is defined as $g(x) = \sqrt{x}$, and $h(x) = g \circ f(x)$.

(c) Write an expression for $h(x)$ and determine the domain and range of $h(x)$. (4 marks)

Solution
$$h(x) = \sqrt{2 - \sqrt{x+5}}$$ Domain: $D_f = \{x \ge -5\}$ and $2 - \sqrt{x+5} \ge 0 \to \sqrt{x+5} \le 2 \to x \le -1$ $$D_h = \{x: x \in \mathbb{R}, -5 \le x \le -1\}$$ Range: $h(-5) = \sqrt{2}$, $h(-1) = 0$. Hence $$R_h = \{y: y \in \mathbb{R}, 0 \le y \le \sqrt{2}\}$$
Specific behaviours
✓ expression for $h(x)$
✓ uses D_f and indicates $R_f \ge 0$
✓ correct domain
✓ correct range

Question 4 **(9 marks)**

(a) Solve the following system of equations and interpret the solution geometrically. (4 marks)

$$x - y + z = 7$$
$$x + 2y + 3z = -10$$
$$x - y - z = 9$$

Solution
$R_1 - R_3: 2z = -2 \to z = -1$ $R_2 - R_1: 3y + 2z = -17 \to y = -5$ $R_1: x + 5 - 1 = 7 \to x = 3$ Solution $x = 3, y = -5, z = -1$ represents the unique point at which the three planes meet.
Specific behaviours
✓ correctly eliminates at least one variable ✓ solves correctly for one variable ✓ solves correctly for all variables ✓ correctly interprets solution

(b) The position vectors of points P, Q and R are $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \overrightarrow{OQ} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $\overrightarrow{OR} = \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}$.

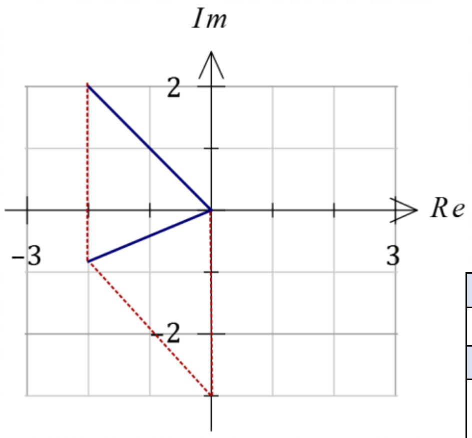
Determine the Cartesian equation of the plane through line PQ and perpendicular to plane OQR. (5 marks)

Solution
Vector perpendicular to plane OQR is $$\overrightarrow{OQ} \times \overrightarrow{OR} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ 7 \\ 5 \end{pmatrix}$$ $$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$$ Hence normal to required plane: $\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \times \begin{pmatrix} -3 \\ 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$ Using point P: $\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} = 13$ Hence Cartesian equation is $4x + y + z = 13$.
Specific behaviours
✓ normal vector to plane OQR ✓ vector $\overrightarrow{PQ}$ ✓ normal to required plane ✓ evaluates constant ✓ writes Cartesian equation

Question 5 **(7 marks)**

Consider the complex number $z = -2 + 2i$.

(a) On the Argand diagram below, draw a line segment from the origin to z and from the origin to $z - 2\sqrt{2}i$. (2 marks)



Solution
See diagram
Specific behaviours
✓ line O to z
✓ line O to $z - \sqrt{2}i$

(b) Determine the principal value of the argument of $z - 2\sqrt{2}i$. (3 marks)

Solution
$z = 2\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$. Using properties of rhombus, the line from O to $z - 2\sqrt{2}i$ bisects angle 2θ between Im axis and line from O to z: $$2\theta = \frac{3\pi}{4}, \qquad \theta = \frac{3\pi}{8}$$ Hence $\arg(z - \sqrt{2}i) = -\frac{\pi}{2} - \frac{3\pi}{8} = -\frac{7\pi}{8}$.
Specific behaviours
✓ indicates polar form of z ✓ uses properties of rhombus   ✓ correct value

(c) Determine the value of the modulus of $z - 2\sqrt{2}i$. (2 marks)

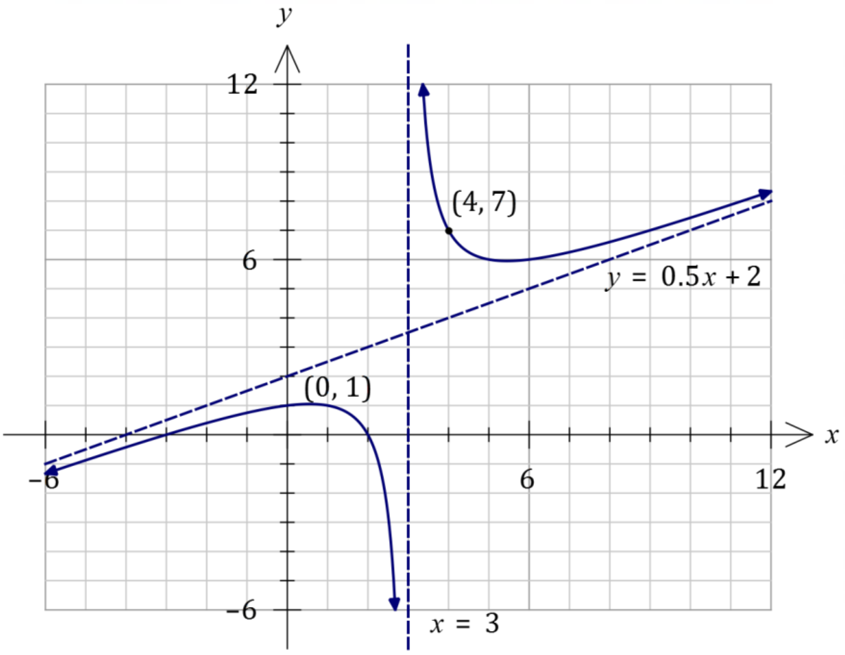
Solution
$$\lvert z - 2\sqrt{2}i \rvert = \sqrt{2^2 + \left(2\sqrt{2} - 2\right)^2}$$ $$= \sqrt{16 - 8\sqrt{2}} = 2\sqrt{4 - 2\sqrt{2}}$$
Specific behaviours
✓ indicates lengths of suitable right triangle   ✓ correct value, with some simplification

Question 6 **(8 marks)**

Consider the function $f(x) = \dfrac{x^2 + bx + c}{ax + d}$, where a, b, c and d are constants.

The graph of $y = f(x)$ has roots at $x = -3$ and $x = 2$, a vertical asymptote $x = 3$ and passes through the point $(4, 7)$.

Sketch the graph of $y = f(x)$, clearly showing the y-intercept and equations of all asymptotes.



Solution
Use roots to determine numerator: $x^2 + bx + c = (x+3)(x-2) = x^2 + x - 6$ Use vertical asymptote to eliminate d: $a(3) + d = 0 \rightarrow d = -3a$ Use point $(4, 7)$ to determine a: $$f(x) = \frac{(x+3)(x-2)}{a(x-3)} \rightarrow 7 = \frac{7 \times 2}{a} \rightarrow a = 2$$ Express $f(x)$ as a proper fraction: $$f(x) = \frac{x^2 + x - 6}{2(x-3)} = \frac{x(x-3)}{2(x-3)} + \frac{4(x-3)}{2(x-3)} + \frac{6}{2(x-3)} = \frac{x}{2} + 2 + \frac{6}{2x-6}$$ Hence oblique asymptote is $y = \frac{x}{2} + 2$ and $f(0) = 1$.

Specific behaviours
✓ uses roots to obtain numerator ✓ uses vertical asymptote to relate a and d ✓ uses point to obtain denominator ✓ expresses $f(x)$ as proper fraction ✓ states correct equation for asymptote ✓ plots roots, y-intercept and both asymptotes ✓ correct curvature of graph to left of vertical asymptote, through roots ✓ correct curvature of graph to right of vertical asymptote, through $(4, 7)$

Question 7 **(7 marks)**

Let the complex number $v = \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$. Describe geometrically the locus of the complex number $z = x + iy$ in the Argand plane that is determined by the relation $\sqrt{2}|z - v^2| = |z - v|$.

Solution
$v = \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right) = -1 - i$ $v^2 = 2\,\text{cis}\left(-\frac{3\pi}{2}\right) = 2i$ $\sqrt{2}\lvert x + iy - (2i)\rvert = \lvert x + iy - (-1 - i)\rvert$ $2(x^2 + (y-2)^2) = (x+1)^2 + (y+1)^2$ $2x^2 + 2y^2 - 8y + 8 = x^2 + 2x + 1 + y^2 + 2y + 1$ $x^2 - 2x + y^2 - 10y = -6$ $(x-1)^2 - 1 + (y-5)^2 - 25 = -6$ $(x-1)^2 + (y-5)^2 = 20 = 2\sqrt{5}$ Hence the locus of z is a circle of radius $2\sqrt{5}$ units with centre at $1 + 5i$.
Specific behaviours
✓ v in Cartesian form ✓ v^2 in Cartesian form ✓ uses $z = x + iy$ and modulus to eliminate i ✓ expands and simplifies equation ✓ factors squared terms ✓ describes locus as a circle ✓ states correct centre and radius of circle

End of questions

Supplementary page

Question number: _________